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On the Stochasticity in Relativistic Cosmology

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It was shown earlier by I. M. Lifshitz and two of us that the evolution of the relativistic cosmological models towards the singularity undergoes spontaneous stochastization.[1] In the present paper it is shown that the statistical parameters of this evolution can be calculated in an exact manner. From the point of view of the general ergodic theory we deal here with a specific mode of stochastization of a deterministic dynamical system with a five-dimensional phase space. The knowledge of the source of stochasticity makes it possible to develop a quantitative statistical theory with appreciable completeness.

KEY WORDS: Stochastization; cosmological models; singularity.

1. INTRODUCTION

The appearance of singularities with respect to time in nonstationary solutions of the Einstein equations is one of the most remarkable features of general relativity; it may well be that its implications are not yet fully appreciated.

There exist different types of singularities in the solutions of the Einstein equations. They can be characterized by indicating three "scale functions" $a(t)$, $b(t)$, $c(t)$ which determine the temporal evolution of the spatial scales in three different directions. It is implied that the solution is formulated in a synchronous reference system, i.e., that the four-dimensional interval $ds^2 = dt^2 - dl^2$, dl being the spatial line element and t the universal time, synchronized over the entire space.

Various types of singularities differ by the degree of their generality. The latter can be measured by the number of "physically arbitrary" functions of the spatial coordinates, which is contained in the broadest class

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of the solutions which admit a singularity of the given type. The general solution should contain such a number of arbitrary functions that any initial conditions (distribution and motion of matter, distribution of the free gravitational field) could be satisfied at a chosen moment of time. This number is four for empty space and eight for space containing matter (see Ref. 2, Sec. 95).

In the widely known homogeneous and isotropic Friedman model the singularity is characterized by the functions $a \sim b \sim c \sim \sqrt{t}$ (singularity is at $t = 0$). However, this type of singularity possesses a low degree of generality; the Friedman solution can be generalized but only to contain merely three physically arbitrary coordinate functions (see Ref. 3, Sec. 4).

Considerably greater is the degree of generality of a singularity of the Kasner type for which

$$a \sim t^{p_a}, \qquad b \sim t^{p_b}, \qquad c \sim t^{p_c} \tag{1.1}$$

where p_a, p_b, p_c are three numbers (the Kasner exponents), satisfying the conditions

$$p_a + p_b + p_c = 1, \qquad p_a^2 + p_b^2 + p_c^2 = 1 \tag{1.2}$$

The class of solutions which possess a singularity of this type contains seven physically arbitrary spatial functions—only one less than is necessary for a general solution (see Ref. 3, Sec. 3).

But most general is the singularity of a complicated oscillatory type. The evolution of this model on approaching the singular point can be described as an infinite succession of interchanging "Kasner epochs" with a certain law of replacement of the Kasner exponents when passing over from one epoch to the next one. The singularity of this type was first discovered for a vacuum homogeneous model of the Bianchi type IX and VIII, and thereupon generalized to the presence of matter. The introduction of matter generates a new property into the evolution of the model: rotation of the Kasner axes [i.e., directions to which the scale functions (1.1) refer] during the interchange of the Kasner epochs, but the law of interchange of the exponents remains the same. The importance of this type of singularities in homogeneous models is due to the fact that it is just the type of singularity in the general cosmological solution of the Einstein equations. The solutions for homogeneous models of the Bianchi types VIII and IX serve as a prototype for construction of the general solution [a review exposition of the relevant results, among them the derivation of the basic rule (2.3) below is given in Refs. 4 and 5).

It is essential that the law of replacement of the Kasner exponents in the oscillatory regime of approach to the singularity in the general inhomogeneous solution remains the same as in homogeneous models. It is just this law that leads to the most important property: spontaneous stochastization of the behavior of the model on approach to singularity and the "loss of memory" of the initial conditions prescribed at some instant of time $t = t_0 > 0$. Thus stochasticity turns out to be a general property of relativistic cosmological models in the neighbourhood of the singularity.

The stochasticity was first pointed out by I. M. Lifshitz and two of us in Ref. 1 (cited henceforth as I). It was shown that the knowledge of the source of stochastization makes it possible to construct with a considerable completeness a statistical theory of the evolution of the cosmological model in asymptotic vicinity to singularity. For a calculation of parameters of this evolution an approximate method was devised, the exactness of which is difficult to estimate beforehand. The aim of the present paper is to shown that these parameters can be calculated exactly (a short account was published in Ref. 6).

In I the theory was developed for a homogeneous model without the rotation of the Kasner axes; this was sufficient for the elucidation of statistical properties ensuing from the law of replacement of the Kasner exponents. The same will be adopted here; we avoid the complications which could be brought about by the rotation of the axes and by inhomogeneity.

The temporal evolution of the vacuum homogeneous model of the Bianchi type IX is governed by the equations

$$2\alpha'' = (b^2 - c^2)^2 - a^4, \qquad 2\beta'' = (a^2 - c^2)^2 - b^4$$
$$2\gamma'' = (a^2 - b^2)^2 - c^4 \tag{1.3}$$

$$\alpha'\beta' + \alpha'\gamma' + \beta'\gamma' - \tfrac{1}{4}(a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2) = 0 \tag{1.4}$$

where along with the functions a, b, c are introduced their logarithms

$$\alpha = \ln a, \qquad \beta = \ln b, \qquad \gamma = \ln c \tag{1.5}$$

the prime denotes differentiation with respct to the variable τ related to the time t by the equation

$$dt = abc\, d\tau \tag{1.6}$$

The Eq. (1.4) contains only the first derivatives and thus plays the role of an additional restriction, imposed on the initial conditions for Eqs. (1.3) (see Ref. 4, Sec. 3). It is easy to verify that the derivative of the expression (1.4) with respect to τ is indeed identically zero due to the Eqs. (1.3); thus if the solution of Eqs. (1.3) satisfies the condition (1.4) in an initial instant of time, the latter will always be satisfied.

From a formal point of view we deal with a deterministic dynamical model, governed by a system of three ordinary differential equations (1.3) [but due to one additional condition (1.4) the phase space of this system is actually not six but five dimensional). Thus, apart from the actual profound cosmological significance of this system, we encounter here a specific mode of spontaneous stochastization of a deterministic system—a phenomenon akin to those discovered recently in many physical problems.

The subject will be presented in such a manner as to allow reading this paper without extensive study of previous work. The basic assertions and previous results needed for understanding will be introduced without derivations, with due references.

2. THE SOURCE OF STOCHASTICITY

Hencefoth we shall denote by p_1, p_2, p_3 (with numerical indices) the Kasner exponents arranged in a fixed order with respect to their magnitude: $p_1 < p_2 < p_3$. These triples of numbers can be parametrized in the form

$$p_1 = -u/f(u), \qquad p_2 = (1+u)/f(u), \qquad p_3 = u(1+u)/f(u)$$
$$f(u) = 1 + u + u^2 \tag{2.1}$$

where the parameter u runs through the values in the region $u \geqslant 1$. On the other hand, values $0 < u < 1$ can be reduced to the same region in view of the formulas

$$p_1(1/u) = p_1(u), \qquad p_2(1/u) = p_3(u), \qquad p_3(1/u) = p_2(u) \tag{2.2}$$

As u decreases from ∞ to 1, the exponent p_1 decreases monotonically, while p_2 and p_3 increase monotonically in the ranges

$$0 \geqslant p_1 \geqslant -1/3, \qquad 0 \leqslant p_2 \leqslant 2/3, \qquad 2/3 \leqslant p_3 \leqslant 1$$

The exponent p_1 is always negative, exponents p_2 and p_3 are positive and it is always $p_3 > p_2$.

The Kasner regime is a solution of Eqs. (1.3)–(1.4) when all the terms in the right-hand sides can be neglected; the time interval during which it is admissible we call a Kasner epoch. Such an interval is certainly cur short with decreasing t since the right-hand sides of Eqs. (1.3) always contain an increasing term. For instance, if the negative exponent refers to the function $a(t)$ ($p_a = p_1$) the perturbation of the Kasner regime will be due to the terms a^4; the remaining terms decrease with decreasing t. This perturbation leads after a brief transitional period to an establishment of a new Kasner epoch with the following rule of replacement of the exponents: if

$$p_a = p_1(u), \qquad p_b = p_2(u), \qquad p_c = p_3(u)$$

then (2.3)

$$p'_a = p_2(u-1), \qquad p'_b = p_1(u-1), \qquad p'_3 = p_3(u-1)$$

where the primed exponents refer to the new epoch [see Ref. 4, Sec. 3; a concise analysis of the system (1.3)–(1.4) and the derivation of the rule (2.3) is also given in Ref. 2, Sec. 118]. The function $a(t)$ acquires a positive exponent and starts to decrease (with decreasing t); the function $b(t)$ acquires a negative exponent and starts to increase; the function $c(t)$ continues to decrease.

The subsequent evolution with the increasing function $b(t)$ leads in an analogous way to the next interchange of the Kasner epochs and so on. The successive interchanges according to the rule (2.3), accompanied by a bouncing of the negative exponent between the functions $a(t)$ and $b(t)$, continues as long as the integral part of the initial value of u is exhausted, i.e., until u becomes less than unity. The value $u < 1$ transforms into $u > 1$ according to (2.2); at this moment either the exponent p_a ot p_b is negative and p_c becomes the smaller one of the two positive exponents ($p_c = p_2$). The next sequence of changes will bounce the negative exponent between the functions c and a or between c and b. For an arbitrary (irrational) initial value of u the process continues indefinitely.

Thus the evolution of the model on approaching the singularity consists of successive periods (we shall call them for brevity "eras") during which two of the scale functions oscillate and the third one decreases monotonically. On passing from one era to another the monotonic decrease is transferred to another of the three scale functions.

To each (sth) era there corresponds a series of values of the parameter u starting with a certain largest one, $u_s^{(\max)}$ and reaching the smallest one, $u_s^{(\min)} < 1$, via the values $u_s^{(\max)} - 1$, $u_s^{(\max)} - 2, \ldots$. We put

$$u_s^{(\max)} = k_s + x_s, \qquad u_s^{(\min)} = x_s \tag{2.4}$$

i.e.,

$$k_s = [u_s^{(\max)}], \qquad x_s = \{u_s^{(\max)}\} \tag{2.5}$$

(the square brackets denote the integer part of a number and the curly brackets denote its fractional part). The number k_s determines the length of the era measured in terms of the number of Kasner epochs is contains. For the next era

$$u_{s+1}^{(\max)} = 1/x_s, \qquad k_{s+1} = [1/x_s] \tag{2.6}$$

The sequence of the lengths of the successive eras has a character of a random process. The source of this stochasticity in just the rule (2.6). This rule states, in other words, that if the entire infinite sequence begins with a certain initial value $u_0^{(\max)} = k_0 + x_0$, then the lengths of the eras $k_0, k_1, k_2,...$ are the numbers in the continuous fraction expansion

$$u_0^{(\max)} = k_0 + \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cdots}} \tag{2.7}$$

This expansion corresponds to the mapping transformation of the interval $[0, 1]$ onto itself by the formula

$$Tx = \{1/x\}, \qquad \text{i.e., } x_{s+1} = \{1/x_s\} \tag{2.8}$$

This transformation belongs to the so-called expanding transformations of the interval $[0, 1]$, i.e., transformations $x \to f(x)$ with $|f'(x)| > 1$. Such transformations possess the property of exponential instability: if we take initially two close points their mutual distance increases exponentially under the iterations of the transformations. It is well known that the exponential instability leads to the appearance of strong stochastic properties.

One can arrive at a probabilistic description of such a sequence by considering not a definite initial value x_0 but the values $x_0 \equiv x$ distributed over the interval $[0, 1]$ in accordance with a certain probability density $w_0(x)$. Then all the subsequent x_s will also be distributed by certain statistical laws $w_s(x)$. Stochastization is manifested in functions $w_s(x)$ tending to a stationary (i.e., independent of s) limiting distribution $w(x)$, which is completely independent of the initial distribution $w_0(x)$ (in general ergodic theory this property is known as mixing (see Ref. 7). The density of this limiting distribution is

$$w(x) = 1/(1+x)\ln 2 \tag{2.9}$$

This formula (which was known already to Gauss) gives the density of the invariant measure of the transformation (2.8).

In order for the sth era to have a length k_s, the preceding era must terminate with a number x in the interval between $1/(1+k)$ and $1/k$.

Therefore the probability for an era to have a length k is equal (in the stationary limit) to

$$W(k) = \int_{1/(1+k)}^{1/k} w(x)\,dx = \frac{1}{\ln 2} \ln \frac{(k+1)^2}{k(k+2)} \tag{2.10}$$

At large values of k

$$W(k) \approx 1/k^2 \ln 2 \tag{2.11}$$

In relating the statistical properties of the cosmological model with the ergodic properties of the transformation (2.8) an important point must be mentioned. In an infinite sequence of numbers x constructed in accordance with this rule, there will be observed arbitrary small (but never vanishing) values of x and accordingly arbitrarilly large lengths k. It was pointed out in Secs. 2 and 3 of Ref. 1, that such cases can (by no means necessarily!) give rise to certain specific situations when the notion of eras, as of sequences of Kasner epochs interchanging each other according to the rule (2.3), loses its meaning (although the oscillatory mode of evolution of the model still persists). Such an "anomalous" situation can be manifested, for instance, in the necessity to retain in the right-hand side of Eqs. (1.3) terms not only with one of the functions a, b, c (say, a^4), as is the case in the "regular" interchange of the Kasner epochs, but simultaneously with two of them (say, a^4, b^4, a^2b^2).

On emerging from an "anomalous" series of oscillations a succession of regular eras is restored. Statistical analysis of the behavior of the model which is entirely based on regular iterations of the transformations (2.6) is corroborated by an important theorem: the probability of the appearance of anomalous cases tends asymptotically to zero as the number of iterations $s \to \infty$ (i.e., the time $t \to 0$). The proof of this assertion was given in Sec. 4 of Ref. 1 and we shall not repeat it here. We merely record that its validity is largely due to a very rapid rate of increase of the oscillation amplitudes during every era and especially in transition from one era to the next one.

We shall be interested here not in the process of the relaxation of the cosmological model to the "stationary" statistical regime (with $t \to 0$ starting from a given "initial instant"), but with the properties of this regime itself with due account taken for the concrete laws of the variation of the physical characteristics of the model during the successive eras.

3. RECURRENCE FORMULAS FOR SUCCESSIVE ERAS

The solution of the problem thus stated is based both on the probability distributions (2.9)–(2.11) and on the formulas describing the variation of the scale functions during successive eras. These variations are subject to certain regularities which become considerably simpler in the asymptotic vicinity to the singularity. We shall repeat here some necessary formulas with a more complete (as compared to Ref. 1, Sec. 2) justification of the adopted approximations.

During each Kasner epoch the product $abc = \Lambda t$ with its own coefficient Λ; correspondingly, $\alpha + \beta + \gamma = \ln \Lambda + \ln t$; we show now that in this equation the constant term $\ln \Lambda$ can be neglected in comparison with $\ln t$.

On changing over from one epoch (with a given value of the parameter u) to the next epoch the constant Λ is multiplied by

$$1 + 2p_1 = (1 - u + u^2)/(1 + u + u^2) < 1$$

(see Ref. 4, Sec. 2). Thus a systematic decrease in Λ takes place. But it is essential that the mean (with respect to the lengths k of eras) value of the entire variation of $\ln \Lambda$ during an era is finite. Actually the divergence of the mean value could be due only to a too rapid increase of this variation with increasing k. For large value of the parameter u we have $\ln(1 + 2p_1) \approx -2/u$. For a large k the maximal value $u^{(\max)} = k + x \approx k$. Hence the entire variation of $\ln \Lambda$ during an era is given by a sum of the form

$$\sum \ln(1 + 2p_1) = \cdots + \frac{1}{k-2} + \frac{1}{k-1} + \frac{1}{k}$$

only the terms which correspond to large values of u are written down here. When k increases this sum increases as $\ln k$. But the probability for an appearance of an era of a large length k decreases as $1/k^2$ according to (2.11); hence the mean value of the sum above is finite. Consequently, the systematic variation of the quantity $\ln \Lambda$ over a large number of eras will be proportional to this number. But it will be shown in the following section [see (5.9)] that with $t \to 0$ the number s increases merely as $\ln |\ln t|$. Thus in the asymptotic limit of arbitrarily small t the term $\ln \Lambda$ can indeed be neglected as compared to $\ln t$. In this approximation we have[3]

$$\alpha + \beta + \gamma = -\Omega \tag{3.1}$$

where Ω denotes the "logarithmic time":

$$\Omega = -\ln t \tag{3.2}$$

Thus the adopted approximation corresponds to neglecting all the quantities whose ratio to $|\ln t|$ tends to zero as $t \to 0$.

[3] Since a, b, c have the dimension of length, their logarithms are defined only up to an additive constant which depends on the choice of the length units; in this sense the equality (3.1) has a conditional meaning corresponding to a certain choice of the zero value of α, β, γ.

The magnitudes of maxima of the oscillating scale functions are also subject to a systematic variation. Denote by $a_{\max}$ and $a'_{\max}$ two successive maxima (of course they belong actually to two different of the functions a, b, c). Then

$$a'_{\max}/a_{\max} = [(u-1)/u]^{1/2}$$

(see Ref. 1, Sec. 2). For $u \gg 1$ we find that $\alpha'_{\max} - \alpha_{\max} \approx -1/2u$. In the same way as it was done above for the quantity $\ln \Lambda$, one can hence deduce that the mean decrease in the height of the maxima during an era is finite and the total decrease over a large number of eras increases with $t \to 0$ merely as $\ln \Omega$. At the same time the lowering of the minima, and by the same token the increase of the amplitude of the oscillations, proceed (as we shall see) proportional to Ω. In correspondence with the adopted approximation we neglect the lowering of the maxima in comparison with the increase of the amplitudes and put $\alpha_{\max} = 0$, $\beta_{\max} = 0$, $\gamma_{\max} = 0$ for the maximal values of all oscillating functions so that the quantities α, β, γ run only through negative values that are connected with one another at each instant of time by the relation (3.1).

Finally in the same approximation we can neglect the widths (in time) of the intermediate regions between the adjacent Kasner epochs, i.e., consider the interchanges of the epochs as instantaneous. Figure 1 shows schematically the course of variation of the functions $\alpha(\Omega)$, $\beta(\Omega)$, $\gamma(\Omega)$ in this approximation during one era and the beginning of the next one; it is composed of straight segments, each of which corresponds to a Kasner epoch. [In this approximation the intervals of the logarithmic time Ω coincide with the intervals of the variable τ, defined by (1.6).]

In what follows we shall discuss statistical properties of the sequence of eras. The index s numbers eras beginning from an arbitrarilly chosen initial one ($s = 0$). The symbol Ω_s denotes the initial instant of the sth era (defined as the instant when the scale function which was monotonically decreasing during the preceding era begins to increase). The initial amplitudes of that pair from among the functions α, β, γ which experiences oscillation in a given era we denote as $\delta_s \Omega_s$; the quantities δ_s (which assume values between 0 and 1) measure these amplitudes in units of the corresponding Ω_s. The recurrence formulas which determine the rules of transition from an era to the next one are[4]

$$\frac{\Omega_{s+1}}{\Omega_s} = 1 + \delta_s k_s \left(k_s + x_s + \frac{1}{x_s}\right) \equiv \exp \xi_s \tag{3.3}$$

$$\delta_{s+1} = 1 - \frac{(k_s/x_s + 1)\delta_s}{1 + \delta_s k_s(k_s + x_s + 1/x_s)} \tag{3.4}$$

[4] In Sec. 4 of Ref. 1 formula (3.4) contained a misprint in the denominator.

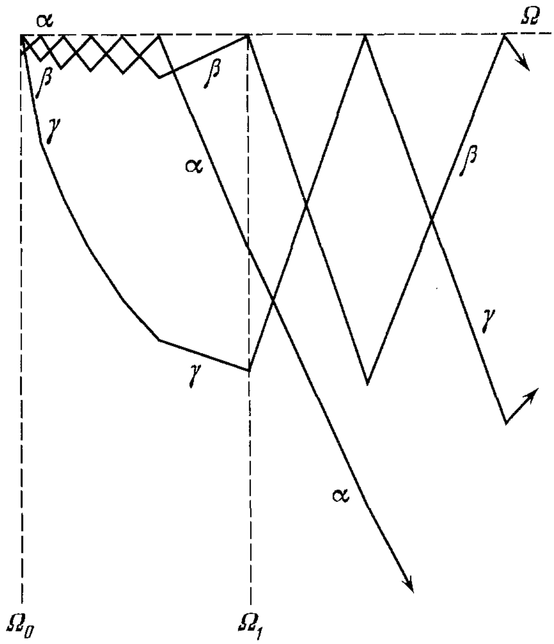

Fig. 1. Schematic plot of evolution of the logarithmic scale functions towards the singularity.

The symbol ξ_s is introduced in (3.3) for future use. Iteration of this formula gives

$$\Omega_s/\Omega_0 = \exp \sum_{p=1}^{s} \xi_p \tag{3.5}$$

4. PROBABILITY DISTRIBUTION OF THE QUANTITIES δ_s

The peculiarity of the statistical properties of the behavior of the system under consideration (the cosmological model) is due to a considerable degree to a comparatively low rate of decrease of the probability distribution $W(k)$ (2.10)–(2.11) for large k. The decrease is so slow that the mean value of k calculated from this distribution diverges logarithmically. If averaging were cut off at a very large but finite value $k = \mathcal{N}$, we would obtain $\langle k \rangle \sim \ln \mathcal{N}$. However, the meaning of a mean value in this case is very limited in view of its instability: the fluctuations of the number k diverge even more rapidly than its mean value. A more adequate statistical characteristics of the sequence of a large number $\mathcal{N}$ of Kasner epochs could be the probability for a randomly picked up epoch to belong to an era of a length $k \leqslant K$, where K is large. This probability equals $\ln K/\ln \mathcal{N}$. It is small if $1 \ll K \ll \mathcal{N}$. In this sense one can say that an epoch randomly chosen from the sequence belongs with a large probability to a long era [in spite of the fact that according to (2.10)–(2.11) the probabilities of the appearance of long eras themselves are small in comparison with the probabilities of short eras, among them of an era with $k=1$).

The quantities δ_s have a stable stationary statistical distribution $P(\delta)$ and a stable (small relative fluctuations) mean value. For their determination in I was used (with due reservations) an approximate method based on the assumption of statistical independence of the random quantity δ_s of the random quantities k_s, x_s. For the function $P(\delta)$ an integral equation was set up which expressed the fact that the quantities δ_{s+1} and δ_s interconnected by the relation (3.4) have the same distribution; this equation was solved numerically. It will be shown now that the distribution $P(\delta)$ can actually be found exactly by an analytical method.

Since we are interested in statistical properties in the stationary limit, it is reasonable to introduce the so-called natural extension of the transformation (2.8) by continuing it without limit to negative indices. Otherwise stated we pass over from a one-sided infinite sequence of the numbers $(x_0, x_1, x_2, \ldots)$, connected by the equalities (2.8), to a "doubly infinite" sequence $X=(\ldots, x_{-1}, x_0, x_1, x_2, \ldots)$ of the numbers which are connected by the same equalities for all $-\infty < s < \infty$. Of course, such expansion is not unique in the literal meaning of the word (since x_{s-1} is not determined uniquely by x_s), but all statistical properties of the extended sequence are uniform over its entire length, i.e., are invariant with respect to arbitrary shift (and x_0 loses its meaning of an "initial" condition). The sequence X is equivalent to a sequence of integers $K=(\ldots, k_{-1}, k_0, k_1, k_2, \ldots)$, constructed by the rule $k_s=[1/x_{s-1}]$. Inversely, every number of X is determined by the integers of K as an infinite continuous fraction

$$x_s = \cfrac{1}{k_{s+1} + \cfrac{1}{k_{s+2} + \cdots}} \equiv x_{s+1}^{+} \tag{4.1}$$

[the convenience of introducing the notation x_{s+1}^{+} with an index shifted by 1 will become clear in the following—cf. (4.4)]. For concise notation we shall denote continuous fraction simply by enumeration (in square brackets) of its denominators; then the definition of x_s^{+} can be written as

$$x_s^{+} = [k_s, k_{s+1}, \ldots] \tag{4.2}$$

We also introduce the quantities which are defined by a continuous fraction with a retrograde (in the direction of diminishing indices) sequence of denominators

$$x_s^- = [k_{s-1}, k_{s-2}, \ldots] \tag{4.3}$$

We transform now the recurrence relation (3.4) by introducing temporarily the notation $\eta_s = (1 - \delta_s)/\delta_s$. Then (3.4) can be rewritten as

$$\eta_{s+1} x_s = 1/(\eta_s x_{s-1} + k_s)$$

By iteration we arrive at an infinite continuous fraction

$$\eta_{s+1} x_s = [k_s, k_{s-1}, \ldots] = x_{s+1}^-$$

Hence $\eta_s = x_s^-/x_s^+$ and finally

$$\delta_s = x_s^+/(x_s^+ + x_s^-) \tag{4.4}$$

This expression for δ_s contains only two (instead of three) random quantities x_s^+ and x_s^-, each of which assumes values in the interval [0, 1].

It follows from the definition (4.3) that $1/x_{s+1}^- = x_s^- + k_s = x_s^- + [1/x_s^+]$. Hence the shift of the entire sequence X by one step to the right means a joint transformation of the quantities x_s^+ and x_s^- accoding to

$$x_{s+1}^+ = \{1/x_s^+\}, \qquad x_{s+1}^- = 1/([1/x_s^+] + x_s^-) \tag{4.5}$$

This is a one-to-one mapping in the unit square. Thus we have now a one-to-one transformation of two quantities instead of a not one-to-one transformation (2.8) of one quantity.

The quantities x_s^+ and x_s^- have a joint stationary distribution $P(x^+, x^-)$. Since (4.5) is a one-to-one transformation, the condition for the distribution to be stationary is expressed simply by a function equation

$$P(x_s^+, x_s^-) = P(x_{s+1}^+, x_{s+1}^-) J \tag{4.6}$$

where J is the Jacobian of the transformation. The normalized solution of this equation is

$$P(x^+, x-) = 1/(1 + x^+ x^-)^2 \ln 2 \tag{4.7}$$

[its integration over x^+ or x^- yields the function $w(x)$ (2.9)].[5] A

[5] The reduction of the transformation to the one-to-one mapping was used already by Chernoff and Barrow[8] and they obtained a formula of the form of (4.7) but for other variables; their paper does not contain applications to the problems which are considered here. As to the preceding papers by Barrow,[9] they contain nothing beyond the main idea (taken from I) about the connection of stochasticity in cosmological models with the transformation (2.8) and the distributions (2.9) and (2.10) (and the repetition of some well-known statements of the general ergodic theory).

constructive derivation of this formula is given in the Appendix. But its correctness can of course be verified also by a direct calculation; the Jacobian of the transformation (4.5) is

$$J = \frac{\partial(x^+_{s+1}, x^-_{s+1})}{\partial(x^+_s, x^+_s)} = \frac{\partial x^+_{s+1}}{\partial x^+_s} \frac{\partial x^-_{s+1}}{\partial x^-_s} = \left(\frac{x^-_{s+1}}{x^+_s}\right)^2$$

(in its calculation one must note that $[1/x_s^+] + \{1/x_s^+\} = 1/x_s^+$).

Since by (4.4) δ_s is expressed in terms of the random quantities x_s^+ and x_s^-, the knowledge of their joint distribution makes it possible to calculate the statistical distribution $P(\delta)$ by integrating $P(x^+, x^-)$ over one of the variables at a constant value of δ. Due to symmetry of the function (4.7) with respect to the variables x^+ and x^- we have $P(\delta) = P(1-\delta)$, i.e., the function $P(\delta)$ is symmetrical with respect to the point $\delta = 1/2$. We have

$$P(\delta)\, d\delta = d\delta \int_0^1 P\left(x^+, \frac{x^+\delta}{1-\delta}\right) \left(\frac{\partial x^-}{\partial \delta}\right)_{x^+} dx^+$$

On evaluating this integral (for $0 \leqslant \delta \leqslant 1/2$ and then making use of the aforementioned symmetry), we obtain finally

$$P(\delta) = 1/(|1 - 2\delta| + 1)\ln 2 \tag{4.8}$$

The solid line in Fig. 2 shows the plot of this function. The dotted line shows the plot of $P(\delta)$ obtained by the approximate method devised in I by

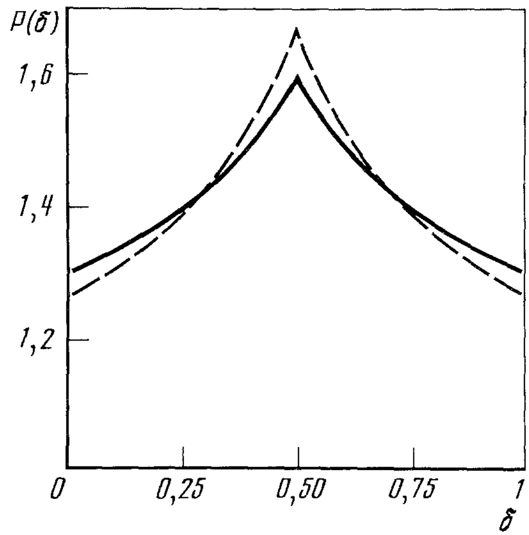

Fig. 2. The probability distribution function $P(\delta)$. Full line: exact function (4.8). Broken line: approximate solution of the integral equation.

numerical solution of an integral equation. Both curves appear to be strikingly similar.[(6)]

The mean value $\langle\delta\rangle = 1/2$ already as a result of the symmetry of the function $P(\delta)$. Thus the mean value of the initial (in every era) amplitude of oscillations of the functions α, β, γ increases as $\Omega/2$.[6]

5. STATISTICAL PARAMETERS OF THE EVOLUTION OF THE MODEL

The expression (3.5) determines the interval of logarithmic time for a succession of a certain number s of eras. A direct averaging of this expression, however, would be meaningless, for the mean values of the quantities $\exp \xi_s$ are unstable in the sense indicated in Sec. 4—the fluctuations increase even more rapidly than the mean value itself with increasing region of averaging. This instability is eliminated by taking the logarithm: the quantities ξ_s have a stable statistical distribution. We denote by τ_s the "double logarithmic" time interval:

$$\tau_s = \ln(\Omega_s/\Omega_0) = \ln|\ln t_s| - \ln|\ln t_0| = \sum_{p=1}^{s} \xi_p \tag{5.1}$$

Its mean value $\langle\tau_s\rangle = s\langle\xi\rangle$.

To calculate $\langle\xi\rangle$ we note that the definition (3.3) can be rewritten as

$$\xi_s = \ln \frac{(k_s + x_s)\delta_s}{x_s(1 - \delta_{s+1})} = \ln \frac{\delta_s}{x_s x_{s-1}(1 - \delta_{s+1})} \tag{5.2}$$

For the stationary distribution $\langle\ln x_s\rangle = \langle\ln x_{s-1}\rangle$, and in virtue of the symmetry of the function $P(\delta)$ also $\langle\ln \delta_s\rangle = \langle\ln(1 - \delta_{s+1})\rangle$. Hence

$$\langle\xi\rangle = -2\langle\ln x\rangle = -2\int_0^1 w(x) \ln x \, dx = \pi^2/6 \ln 2 = 2.37$$

[$w(x)$ from (2.9)]. Thus

$$\langle\tau_s\rangle = 2.37s \tag{5.3}$$

[6] The plot of the function $P(\delta)$ in Fig. 2 in I is incorrect for several reasons. Apparently some errors were admitted in preparing the program for numerical solution of the integral equation. Also a "forced" reduction of the values $P(0)$ and $P(1)$ was performed in view of the incorrect footnote in I, Sec. 4. It is to be emphasized that the finite probability of the value $\delta = 0$ does not mean the possibility of the initial amplitude of oscillation becoming zero (which would be in contradiction to the regular course of evolution shown in Fig. 1). Indeed it is seen from (3.4) that δ_{s+1} tends to zero with $x_s \to 0$ proportional to x_s; but the amplitude is given by the product $\delta_{s+1}\Omega_{s+1}$, which tends to a finite limit since the expression (3.3) contains a term with $1/x_s$.

For large s the number of terms in the sum (5.1) is large and according to general theorems of the ergodic theory the values of τ_s are distributed around $\langle\tau_s\rangle$ according to Gauss' law with the density

$$\rho(\tau_s) = (2\pi D_\tau)^{-1/2} \exp[-(\tau_s - \langle\tau_s\rangle)^2/2D_\tau] \tag{5.4}$$

Calculation of the variance D_τ is more complicated since not only the knowledge of $\langle\xi\rangle$ and $\langle\xi^2\rangle$ are needed but also of the correlations $\langle\xi_p\xi_{p'}\rangle$. The calculation can be simplified by rearranging the terms in the sum (5.1). By using (5.2) we rewrite this sum as

$$\begin{aligned}\sum_{p=1}^{s} \xi_p &= \ln \prod_{p=1}^{s} \frac{\delta_p}{(1-\delta_{p+1})x_p x_{p-1}} \\ &= \ln \prod_{p=1}^{s} \frac{\delta_p}{(1-\delta_p)x_{p-1}^2} + \ln\frac{x_0}{x_s} + \ln\frac{1-\delta_1}{1-\delta_{s+1}}\end{aligned}$$

The last two terms do not increase with increasing s; being interested in the limiting laws for large s we can omit these terms. Then

$$\sum_{p=1}^{s} \xi_p = \sum_{p=1}^{s} \ln(1/x_p^+ x_p^-) \tag{5.5}$$

[here also the expression (4.4) for δ_p is taken into account]. We notice that to the same accuracy (i.e., up to the terms which do not increase with s) the equality

$$\sum_{p=1}^{s} \ln x_p^+ = \sum_{p=1}^{s} \ln x_p^- \tag{5.6}$$

is valid. Indeed in virtue of (4.5) we have

$$x_{p+1}^+ + 1/x_{p+1}^- = 1/x_p^+ + x_p^-$$

and hence

$$\ln(1 + x_{p+1}^+ x_{p+1}^-) - \ln x_{p+1}^- = \ln(1 + x_p^+ x_p^-) - \ln x_p^+$$

By summing this identity over p we obtain (5.6). Finally we change (again with the same accuracy) x_p^+ for x_p under the summation sign and thus represent τ_s as

$$\tau_s = \sum_{p=1}^{\infty} \eta_p, \qquad \eta_p = -2\ln x_p \tag{5.7}$$

The variance of this sum in the limit of large s is

$$D_{\tau_s} = \langle(\tau_s - \langle\tau_s\rangle)^2\rangle \approx s \left\{ \langle\eta^2\rangle - \langle\eta\rangle^2 + 2 \sum_{p=1}^{\infty} (\langle\eta_0\eta_p\rangle - \langle\eta\rangle^2) \right\} \quad (5.8)$$

It is taken into account here that in virtue of the statistical homogeneity of the sequence X the correlations $\langle\eta_p\eta_{p'}\rangle$ depend only on the differences $|p - p'|$. The mean value $\langle\eta\rangle = \langle\xi\rangle$; the mean square

$$\langle\eta^2\rangle = 4 \int_0^1 w(x) \ln^2 x \, dx = 6\zeta(3)/\ln 2 = 10.40$$

By taking into account also the values of correlations $\langle\eta_0\eta_p\rangle$ with $p = 1, 2, 3$ (calculated numerically) we arrive at the final result $D_{\tau_s} = (3.5 \pm 0, 1)s$.

With increasing s the relative fluctuation $D_{\tau_s}^{1/2}/\langle\tau_s\rangle$ tends to zero as $s^{-1/2}$. In other words, the statistical relation (5.3) becomes almost certain at large s. This makes it possible to invert the relation, i.e., to represent it as the dependence of the average number of the eras s_τ that are interchanged in a given interval τ of the double logarithmic time:

$$\langle s_\tau\rangle = 0.42\tau \quad (5.9)$$

The statistical distribution of the exact values of s_τ around its average is also Gaussian with the variance

$$D_{s_\tau} = 3.5\langle s_\tau\rangle^3/\tau^2 = 0.26\tau$$

It was already mentioned in Sec. 1 that the source of stochasticity of the model—the rule of interchange of the Kasner exponents—remains the same upon introduction of matter. All the results pertaining the evolution of the matter density which were formulated in Sec. 4 of I remain unaltered and we shall not repeat them here.

DEDICATION

The basic work I was accomplished in active cooperation with Il'ya M. Lifshitz. He was not only a distinguished theoretical physicist but also an excellent mathematician. His profound insight into statistical problems was invaluable in formulation of the basic statements of the above theory. The authors wish to dedicate this paper to his memory.

APPENDIX

A constructive derivation of the distribution (4.7) is given here.

A shift of the sequence X by one step gives rise to the following transformation T of the unit square:

$$x' = \{1/x\}, \qquad y' = 1/([1/x] + y)$$

[with $x \equiv x_0^+$, $y \equiv x_0^-$, cf. (4.5)]. The density $P(x, y)$ defines the invariant measure for this transformation. It is natural to suppose that $P(x, y)$ is a symmetric function of x and y. This means that the measure is invariant with respect to the transformation $S(x, y) = (y, x)$ and hence with respect to the product ST. Evidently $ST(x, y) = (x'', y'')$ with

$$x'' = 1/([1/x] + y), \qquad y'' = \{1/x\}$$

Evidently ST has a first integral $H = 1/x + y$. On the line $H = \text{const} \equiv c$ the transformation has the form

$$\frac{1}{x''} = \left[\frac{1}{x}\right] + y = \left[\frac{1}{x}\right] + c - \frac{1}{x} = c - \left\{\frac{1}{x}\right\}$$

Hence the invariant measure density of ST must be of the form

$$f(c)\, dc\, d\frac{1}{x} = f\left(\frac{1}{x} + y\right) \frac{1}{x^2}\, dx\, dy$$

With the account taken of the symmetry $P(x, y) = P(y, x)$ we get $f(c) = c^{-2}$ and hence (after normalization) the result (4.7).

NOTE ADDED IN PROOF

We were recently informed by D. E. Chernoff that the formula (4.8) (for the probability distribution introduced in Ref. 1) was independently derived by him in an unpublished essay.

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